

# TSIRELSON'S PROBLEM AND PURELY ATOMIC VON NEUMANN ALGEBRAS

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ABSTRACT. It is shown that if a bipartite behavior admits a field representation in which Alice (or Bob's) observable algebra generates a purely atomic von Neumann algebra then it is non-relativistic.

Let  $H$  be a separable complex Hilbert space, and let  $B(H)$  be the algebra of all bounded linear operators on  $H$ . If  $S \subset B(H)$  then  $\text{span}(S)$  denotes its linear span and  $\text{comm}(S)$  its commutant.

Let  $C_1(H)$  be the space of all trace-class operators on  $H$ . We denote

$$\langle \mu, T \rangle = \text{tr}(\mu T) \quad \mu \in C_1(H), T \in B(H).$$

If  $\Psi : B(H) \rightarrow B(H)$  is a weak star continuous map, then we shall denote by  $\Psi^* : C_1(H) \rightarrow C_1(H)$  its predual map, hence

$$\langle \Psi^*(\mu), T \rangle = \langle \mu, \Psi(T) \rangle \quad \mu \in C_1(H), T \in B(H).$$

In what follows  $A$  and  $B$  are finite sets. Moreover  $\{A_x\}_{x \in A}$  and  $\{B_y\}_{y \in B}$  are families of finite sets. Elements of  $A_x$  are identified by pairs of the form  $(a, x)$  and similarly for  $B_y$ .

The following result has been recently proved in [4].

**Theorem 1.** *Let  $\{E_a^x\}_{a,x}$  and  $\{F_b^y\}_{b,y}$  be two families of positive operators in  $B(H)$  such that*

- (i)  $\sum_a E_a^x = 1 \quad (\forall) x \in A$
- (ii)  $\sum_b F_b^y = 1 \quad (\forall) y \in B$
- (iii)  $E_a^x F_b^y = F_b^y E_a^x \quad (\forall) a, b, x, y.$

*Let  $\rho \in C_1(H)$  be positive with  $\text{tr}(\rho) = 1$  and let*

$$p(a, b|x, y) = \langle \rho, E_a^x F_b^y \rangle.$$

*Suppose there exist a family  $\{\sigma_a^x\}_{a,x}$  of positive trace-class operators on  $H$  such that*

- (iv)  $\sigma = \sum_a \sigma_a^x$  does not depend on  $x \in A$  and
- (v)  $\langle \sigma_a^x, F_b^y \rangle = p(a, b|x, y)$  for all  $a, b, x, y.$

*Then there exist families  $\{\tilde{E}_a^x\}_{a,x}$  and  $\{\tilde{F}_b^y\}_{b,y}$  of positive operators on  $H$  and a normal state  $\tilde{\rho}$  on  $B(H \otimes H)$  such that*

$$\sum_a \tilde{E}_a^x = 1 \quad (\forall) x \in A$$

and

$$\sum_b \tilde{F}_b^y = 1 \quad (\forall) y \in B$$

and such that

$$p(a, b|x, y) = \langle \tilde{\rho}, \tilde{E}_a^x \otimes \tilde{F}_b^y \rangle \quad (\forall) a, b, x, y.$$

This result is related to a certain problem in the theory of quantum correlations formulated in [6] and [7]. For recent work in this area we refer to [2], [3],[4],[5] and the references therein. In this paper we shall provide a class of examples where this theorem applies.

**Proposition 2.** *Suppose  $\{E_a^x\}_{a,x}$  and  $\{F_b^y\}_{b,y}$  are families of positive operators on  $H$  satisfying (i)-(iii) in Theorem 1 and let  $\rho \in C_1(H)$  be positive with  $\text{tr}(\rho) = 1$ .*

*Assume there exists a positive linear weak star continuous idempotent map*

$$\Phi : B(H) \rightarrow B(H)$$

such that

$$\text{span}(\{F_b^y\}_{b,y}) \subset \text{range}(\Phi) \subset \text{comm}(\{E_a^x\}_{a,x})$$

*Then there exist a family  $\{\sigma_a^x\}_{a,x}$  of positive trace-class operators on  $H$  such that (iv) and (v) in Theorem 1 hold true.*

*Proof.* Let us define

$$\sigma_a^x = \Phi^*((E_a^x)^{1/2} \rho (E_a^x)^{1/2})$$

for all  $a, x$ . Then for every  $T \in B(H)$  we have

$$\langle \sum_a \sigma_a^x, T \rangle = \langle \rho, \sum_a (E_a^x)^{1/2} \Phi(T) (E_a^x)^{1/2} \rangle = \langle \rho, \Phi(T) \rangle = \langle \Phi^*(\rho), T \rangle$$

therefore  $\sigma = \sum_a \sigma_a^x$  does not depend on  $x$ . Moreover for every  $a, x, b, y$  we have

$$\langle \sigma_a^x, F_b^y \rangle = \langle \rho, (E_a^x)^{1/2} \Phi(F_b^y) (E_a^x)^{1/2} \rangle = \langle \rho, E_a^x F_b^y \rangle = p(a, b|x, y).$$

□

This proof is in part inspired by the proof of Thm 5 in [4]. Recall that a purely atomic von Neumann algebra is one in which the identity is a sum of minimal projections. Obviously, every finite dimensional von Neumann algebra is purely atomic as well as  $B(H)$  itself. It is known that any such algebra is the range of a weak star continuous completely positive idempotent. It follows that the above result applies for instance when either  $\{E_a^x\}_{a,x}$  or  $\{F_b^y\}_{b,y}$  generate purely atomic von Neumann algebras. For terminology and results on this class of algebras we refer to [1].

## REFERENCES

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